NONSTATIONARY FLOW OF A CONDUCTING GAS IN CROSSED ELECTRIC AND MAGNETIC FIELDS

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The plasma is inviscid, cool, and not thermally conducting; it flows in a channel of constant cross section. The solution is derived by the small-parameter method, for which purpose the magnetic interaction N is used. There have been previous studies of the transient-state flow of an inviscid and thermally nonconducting plasma in crossed electric and magnetic fields [1–3]. A plasma of infinite conductivity has been considered [1], as well as flow involving entropy change in an MHD system with strong electromagnetic fields [2, 3].

Consider the one-dimensional nonstationary flow of an inviscid plasma that conducts electricity but not heat, which has a small magnetic Reynolds number ($R_m \ll 1$), in a channel of constant cross section in an MHD converter.

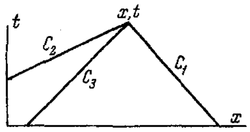

Fig. 1

The system of dimensionless equations for magnetic gasdynamics takes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0, \quad \rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial x} = -\frac{\partial p}{\partial x} - NjB,$$

$$\rho \frac{\partial}{\partial t}\left(\frac{k}{k-1}\frac{p}{\rho} + \frac{v^2}{2}\right) + \rho v \frac{\partial}{\partial x}\left(\frac{k}{k-1}\frac{p}{\rho} + \frac{v^2}{2}\right) =$$

$$= \frac{\partial p}{\partial t} - NjU,$$

$$j = \sigma(vB - U), \quad \sigma = \sigma(p, \rho), \quad N = \sigma_0 L B_0^2 / \rho_0 v_0, \qquad (1)$$

in which N is the parameter for the magnetic interaction. The dimensionless quantities in (1) are introduced via the following relations:

$$x = \frac{x'}{L}, \quad y = \frac{y'}{L} = 1, \quad b = \frac{b'}{L}, \quad l = \frac{l'}{L},$$

$$v = \frac{v'}{v_0}, \quad \rho = \frac{\rho'}{\rho_0}, \quad p = \frac{p'}{\rho_0 v_0^2}, \quad t = \frac{t' v_0}{L},$$

$$B = \frac{B'}{B_0}, \quad \sigma = \frac{\sigma'}{\sigma_0}, \qquad T = T'/T_0,$$

$$j = j'/\sigma_0 v_0 B_0, \quad U = U'/LB_0 v_0, \quad R = R'\sigma_0 L, \qquad (2)$$

in which R is the resistance of the load, while the other symbols are as usual. The subscript 0 denotes quantities at the entrance to the channel.

In the general case, system (1) must be supplemented by relationships for the compressor, the exit system, and the total current I as a function of load voltage.

Here we consider only (1), with the following initial and boundary conditions:

$$p = p(x), \quad v = v(x), \quad \rho = \rho(x) \text{ for } t = 0,$$

$$p = F_1(t), \quad \rho = F_2(t) \text{ for } x = 0,$$

$$p = F_3(t) \text{ for } x = l, \qquad (3)$$

in which p(x), v(x), and ρ(x) are solutions to the steady-state problem, while $F_1(t)$, $F_2(t)$, and $F_3(t)$ are given functions.*

If B = const, the voltage U for continuous electrodes is a function of time only, U = U(t); but if B = B(t), we cannot have U(t) as an arbitrary time function, since it is dependent on B and on the load resistance R.

We use the equation of motion to transform the energy equation and also use Ohm's law to eliminate the current density from (1). Then

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v}{\partial x} + v \frac{\partial \rho}{\partial x} = 0,$$

$$\rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial x} + \frac{\partial p}{\partial x} = -N\sigma B(vB - U),$$

$$v \frac{\partial p}{\partial x} + \frac{\partial p}{\partial t} - k\frac{p}{\rho}\frac{\partial \rho}{\partial t} - k\frac{p}{\rho} v \frac{\partial \rho}{\partial x} =$$

$$= N(k-1)\sigma(vB - U)^2, \quad \sigma = \sigma(p, \rho). \qquad (4)$$

System (4) contains the parameter N, which is small (N < 1) in many cases of practical interest. We therefore seek the solution as series expansions of the unknown functions in powers of N [4]:

$$z = z_0(x, t) + Nz_1(x, t) + N^2 z_2(x, t) + \ldots, \qquad (5)$$

in which z is p, ρ, or v, z_1 is p_1, ρ_1, or v_1, ..., and $z_0(x,t)$ is the solution of the nonstationary problem for zero field.

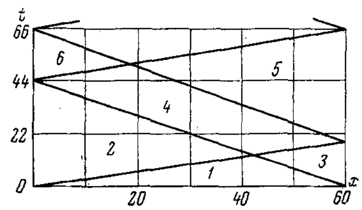

Fig. 2

*The form of the boundary conditions is dependent on the problem; for example, we may be given the gas flow rate G = G(t), i.e., the relation of velocity to density, or alternatively, the relation between the pressure, velocity, and density, or other relations between the parameters of the gas at the inlet and outlet.

Consider the flow in a channel when the gas parameters at inlet and outlet are not dependent on time, i.e., F_1, F_2, and F_3 in (3) are constants. Then

$$p_0(x, t) = p_0 = \text{const},$$

$$\rho_0(x, t) = 1, \qquad v_0(x, t) = 1.$$

Substitution of (5) into (4) and matching of coefficients for identical powers of N gives us a set of linear systems of first-order differential equations for the functions $z_1, z_2, \ldots, z_n$. For the z_1 we have

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial \rho_1}{\partial x} + \frac{\partial v_1}{\partial x} = 0,$$

$$\frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x} + \frac{\partial p_1}{\partial x} = -\sigma B(B - U),$$

$$\frac{\partial}{\partial t}(p_1 - kp_0\rho_1) +$$

$$+ \frac{\partial}{\partial x}(p_1 - kp_0\rho_1) = \sigma(k-1)(B-U)^2,$$

$$\sigma = \sigma(p, \rho). \qquad (6)$$

For the z_n we may derive an analogous system in whose right-hand part U and B are accompanied by $z_1, \ldots, z_{n-1}$ and derivatives of these, so the systems may be solved successively.

Linear transformation of the unknown functions [5] gives

$$p_1 = a_0(w_2 - w_1), \qquad v_1 = w_2 + w_1,$$

$$\rho_1 = a_0^{-1}(w_2 - w_1) - a_0^{-2} w_3, \qquad (7)$$

and the system of equations acquires the canonical form

$$\partial w_i / \partial t + \lambda_i \partial w_i / \partial x = f_i \qquad (i = 1, 2, 3),$$

$$f_1 = -\frac{\sigma(B-U)}{2}\left[\frac{k-1}{a_0}(B-U) + B\right],$$

$$f_2 = \frac{\sigma(B-U)}{2}\left[\frac{k-1}{a_0}(B-U) - B\right],$$

$$f_3 = \sigma(k-1)(B-U)^2, \qquad \lambda_1 = 1 - a_0,$$

$$\lambda_2 = 1 + a_0, \quad \lambda_3 = 1 \qquad (a_0 = \sqrt{kp_0}), \qquad (8)$$

in which a_0 is the speed of sound at the entry to the channel. As all the λ_i are real numbers, system (8) is a hyperbolic one, and through each point in the xt-plane pass the three real characteristics (Fig. 1) defined by

$$dx/dt = 1 - a_0 \ (C_1),$$

$$dx/dt = 1 + a_0 \ (C_2), \qquad dx/dt = 1 \ (C_3). \qquad (9)$$

System (8) is equivalent [6] to a system of integral equations

$$w_i(x, t) = w_i(x_i, 0) + \int_{C_i} f_i dt \qquad (10)$$

or

$$w_i(x, t) = w_i(0, t_i) + \int_{C_i} f_i dt,$$

in which C_i is the part of the corresponding characteristic from (x, t) to the intersection at $(x_i, 0)$ or $(0, t_i)$ with the x- or t-axis, respectively (Fig. 1). The functions $w_i(x_i, 0)$ or $w_i(0, t_i)$ are deduced from the initial or boundary conditions.

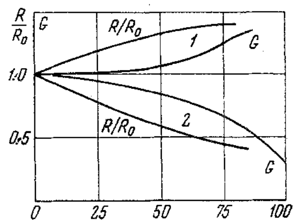

Fig. 3

If the electrical conductivity σ = const, while U(t) is an analytic function, then system (10) can be integrated, which solves the problem, because p, ρ, and v are found via (7) and (5).

If the change in conductivity cannot be neglected, system (10) is solved by successive approximation, with the zeroth approximation taken as $\sigma(x)$, the result for the steady-state problem. The method of finite differences [5] may also be used to solve system (8).

As an example we consider the case in which B and σ are constant, while

$$U = U_0 + \alpha t. \qquad (11)$$

We assume that the pressure and density (temperature) at the inlet are constant, while at the exit there is a constant pressure equal to the pressure p_* for $U = U_0$.

We use the boundary conditions of (3) and expansion (5) with

$$p_* = p + {}^1/_2\, \rho v^2 \,(1 + {}^1/_4\, \rho v^2 / kp) \qquad (12)$$

to get the boundary conditions for this problem as

$$p_1 = 0, \quad \rho_1 = 0 \quad \text{for } x = 0,$$

$$p_{1*} = \beta p_1 + \delta \rho_1 + \gamma v_1 \quad \text{for } x = l,$$

$$p_1 = p_1(x), \quad v_1 = v_1(x), \quad \rho_1 = \rho_1(x) \quad \text{for } t = 0 \qquad (13)$$

in which $p_1(x)$, $\rho_1(x)$, and $v_1(x)$ are derived from the solution to the steady-state problem for $U = U_0$, while

$$\beta = 1 - \frac{1}{8a_0^2 p_0}, \qquad \delta = \frac{1}{2} + \frac{1}{4a_0^2}, \qquad \gamma = 1 + \frac{1}{2a_0^2}.$$

It follows from (7) that

$$w_1 = \frac{1}{2}\left(v_1 - \frac{p_1}{a_0}\right), \quad w_2 = \frac{1}{2}\left(v_1 + \frac{p_1}{a_0}\right),$$

$$w_3 = p_1 - a_0^2 \rho_1. \qquad (14)$$

Then system (13) allows us to derive the initial and boundary conditions for w_3 as well as the initial conditions for w_1 and w_2. It is more complicated to derive the boundary conditions for w_1 and w_2. Figure 2 shows the regions formed by families C_1 and C_2 of characteristics in the xt-plane.

The function $w_2(x_i, 0)$ is derived via the initial conditions in integrating (10) in regions 1–3; $w_1(x_i, 0)$ in regions 1–2 is derived similarly. The boundary conditions at x = 0 for $w_2(0, t_j)$ in regions 2, 4, and 5 are derived via (7) and (13) together with the known values of w_1 in region 1–2 with x = 0. The boundary conditions at $x = l$ for w_1 in regions 3, 4, and 6 are derived similarly, and so on.

After integration of (10) throughout the region of interest via (5) and (7), we derive v(x, t), p(x, t), and ρ(x, t) and calculate the total current, the load resistance, and the gas flow rate from

$$I(t) = b\int_0^l j(x, t)\,dx, \qquad G(x, t) = \rho(x, t)\,v(x, t),$$

$$R(t) = U(t)\left\{b\int_0^l [v(x, t) - U(t)]\,dx\right\}^{-1}.$$

Figure 3 shows results for two modes of variation of U with $U_0 = 0.5$: 1) $\alpha_1 = 0.003$, 2) $\alpha_2 = -0.004$. The characteristic parameters used in (2) were

$$L = 0.05\,\text{m}, \quad b' = 0.05\,\text{m}, \quad l' = 3\,\text{m},$$

$$v_0' = 434\,\text{m/sec}, \qquad T_0' = 3023^\circ\,\text{K}, \quad B_0' = 1.5\,\text{Wb/m}^2,$$

$$\sigma_0' = 100\ \text{ohm}^{-1}/\text{m}^{-1}, \qquad p' = 10^6\ \text{N/m}^2,$$

$$U_0' = 16.2\,\text{V}, \quad N = 0.0225.$$

It is clear that a linear variation in U requires essentially nonlinear variation in R, and the variation in G is also nonlinear. The results show that there is hysteresis in linear variation from U_0 to U and return from U to U_0 by the same law, i.e., to a given voltage there correspond different values of the velocity, pressure, gas flow rate, and drawn power.

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